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LETTER TO THE EDITOR

Non-universal critical behaviour in planar Ising models with extended, radially symmetric defects

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Abstract. Using conformal mapping and finite-size scaling we determine the local magnetic exponent near a defect characterized by couplings of the form $J(r) = J(1+A/r)$). In accordance with phenomenological considerations, we find that it depends continuously on the amplitude A, both for a defect in the interior and on the boundary.

The theory of conformal invariance classifies homogeneous two-dimensional critical systems and determines their critical exponents and correlation functions without and with surfaces [1, 2]. But conformal invariance also holds when the translational symmetry is broken by a straight defect line [3, 4] or a boundary near which the system is inhomogeneous on a large scale [5]. These two cases are particularly interesting since it has been shown that they have non-universal exponents [6–11]; also the validity of conformal mapping for such a situation has been demonstrated [3–5].

In this letter we study another type of inhomogeneous system which is physically quite reasonable but has not been treated so far. We consider an Ising square lattice with the Hamiltonian

$$H = -\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (J_1(r)\sigma_{mn}\sigma_{mn+1} + J_2(r)\sigma_{mn}\sigma_{m+1,n}) \tag{1}$$

where $\sigma_{m,n} = \pm 1$ and the interaction constants $J_{1,2}(r) = J_{1,2}(1+A(r))$ depend on the distance $r = (m^2+n^2+1)^{1/2}$ from a centre. For $A(r)$ we will assume the form $A(r) = A/r^{\alpha}$. Then, for $\alpha \to \infty$ we have a lattice with a point-like defect, while for $\alpha \to 0$ the defect extends over the whole system. The case $\alpha = 1$ is particular: a phenomenological approach [12] then predicts that locally, i.e. near the centre of the defect, the system shows non-universal behaviour that manifests itself in the fact that the critical exponents are continuous functions of the microscopic parameter A. We test and verify this prediction, using conformal mapping and finite-size calculations. We calculate the exponent of the local order parameter and extend the considerations also to the case where the defect is centred on a boundary and for which the phenomenological theory has not yet been worked out.

We first consider an infinite system with the centre of the defect at the origin. Following Cardy [13] we use the conformal transformation

$$w(z) = \frac{L}{2\pi} \ln z \tag{2}$$

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to map the complex z-plane onto a strip $w = u + \mathrm{i}v$, $-\infty < u < \infty$, $0 \leq v \leq L$ with periodic boundary conditions. The inhomogeneity $A(z)$ then transforms according to [5]

$$
\begin{aligned}
A(w) &= |w'(z)|^{-y_t} A(z) \\
&= \left(\frac{2\pi}{L}\right)^{y_t} \exp[(y_t - \alpha) 2\pi u / L] A
\end{aligned}
\tag{3}
$$

where y_t is the scaling dimension of the temperature variable which equals one in our case. In general, this $A(w)$ describes a system with couplings depending on the position along the strip. For $\alpha = y_t = 1$, however, the system becomes homogeneous with $K_i = J_i/k_{\mathrm{B}}T$ given by

$$
K_i = K_{ic}\left(1 + \frac{2\pi A}{L}\right) \tag{4}
$$

where K_{ic} is the value of K_i at the critical point $T = T_c$. Thus, starting from a defect problem in the plane we arrive at a problem without defect in the strip. This is in contrast to the cases studied so far.

Depending on the sign of A, the system in the strip is slightly above or below the bulk critical temperature by an amount of order $1/L$. A similar situation occurs if one maps a system with inhomogeneous boundary on a strip [5]. Then, however, the couplings are non-uniform across the strip.

Conformal invariance implies a correlation length $\xi \sim L$ in the strip and gives the relation

$$
x_l = \frac{1}{2\pi\zeta} \lim_{L\to\infty} \left(\frac{L}{\xi}\right)
$$

for the anomalous dimension x_l of the local order parameter [13], where ζ is an anisotropy parameter and in our case $\zeta = (\sinh 2K_2/\sinh 2K_1)^{1/2}$ [14]. The scaling dimension x_l determines the behaviour of the local magnetization near the defect $\langle\sigma_0\rangle \sim \tau^{\beta_l}$, $\beta_l = \nu x_l$ and in our case $\nu = 1$ [15, 16]. In order to test this, we consider a lattice with L sites across the strip and calculate ξ from the gap in the spectrum of the transfer matrix $T = \exp(-H)$, where H is bilinear in fermions. As is well known, the gap is the difference between the two fermionic ground state energies of H with periodic and antiperiodic boundary conditions, respectively [15, 16]. Thus,

$$
\xi^{-1} = \begin{cases} D & \text{for } A > 0 \ (T < T_c) \\ D + \gamma(0) & \text{for } A < 0 \ (T > T_c) \end{cases} \tag{5}
$$

where D is the difference

$$
D = \frac{1}{2} \sum_{n=0}^{L-1} \left\{ \gamma\left[\frac{2\pi}{L}\left(n + \frac{1}{2}\right)\right] - \gamma\left(\frac{2\pi}{L} n\right) \right\} \tag{6}
$$

and the single-fermion eigenvalues $\gamma(q) \geq 0$ are given in general by [15]

$$
\cosh \gamma(q) = \cosh 2K_1^* \cosh 2K_2 - \sinh 2K_1^* \sinh 2K_2 \cos q \tag{7}
$$

with $\sinh 2K_\alpha^* = 1/\sinh 2K_\alpha$. If c_l is the lth Fourier coefficient of $\gamma(q)$

$$
c_l = \frac{1}{\pi} \int_{-\pi}^{\pi} \gamma(q) \cos(lq) \, \mathrm{d}q \tag{8}
$$

one has [17]

$$D = -L \sum_{j=0}^{\infty} c_{(2j+1)L}. \tag{9}$$

An analysis of (8) shows that, in the limit $L \to \infty$, the largest contribution comes from small values of the momentum q and one can approximate

$$\gamma(q) \approx \zeta \sqrt{\left(\frac{a}{L}\right)^2 + q^2} \qquad a = 4\pi(K_{1c} + \sinh 2K_{1c}K_{2c})A. \tag{10}$$

Using the change of variable $q = (a/L)\sinh u$, extending the limits of integration with respect to u to $\pm\infty$ and using the result in (9) finally gives $D = I(a)/L$, where

$$I(a) = \int_0^{\infty} \frac{\sinh^2 u \, du}{\sinh(|a| \cosh u)}. \tag{11}$$

Thus D is, in fact, proportional to $1/L$ and the exponent β_l is given by

$$\beta_l(a) = \begin{cases} \dfrac{a^2}{2\pi^2} I(a) & a > 0 \\ \dfrac{a^2}{2\pi^2} I(a) + \dfrac{a}{2\pi} & a < 0. \end{cases} \tag{12}$$

Limiting forms are

$$\beta_l(a) = \begin{cases} \frac{1}{8} - (a/4\pi) + O(a^2) & |a| \ll 1 \\ \sqrt{(a/2\pi^3)}\, e^{-a}[1 + O(a^{-1})] & a \gg 1 \\ (|a|/2\pi) + O(e^{-a}) & a \ll -1. \end{cases} \tag{13}$$

The result for small a agrees with the prediction of the phenomenological approach, in which β_l was calculated in first-order perturbation theory [12]. A plot of the function $\beta_l(a)$ is shown in the figure.

One can repeat these considerations for a half-infinite system. In this case the centre of the defect is placed on the (straight) boundary. The transformation $w = (L/\pi) \ln z$ then maps this system onto a strip with *free* boundary conditions and couplings $K_i = K_{ic}(1 + \pi A/L)$. The deviation from criticality in the strip is now only half as large as before.

The correlation length in this case is obtained from the smallest $\gamma(q)$ where the momenta q have to be determined from the equation [11, 18]

$$\frac{\sinh 2K_1^* \sin q}{\cosh 2K_1^* \sinh 2K_2 - \sinh 2K_1^* \cosh 2K_2 \cos q} = \tan(Lq). \tag{14}$$

The smallest $\gamma(q)$ has $q \sim 1/L$ and one can simplify the left-hand side of (14). The equation then becomes, with $x = Lq$

$$\frac{2x}{a} = \tan x \tag{15}$$

and has the smallest real solution in the interval $0 \leq x \leq \pi$ for $a/2 \leq 1$. For $a/2 > 1$, x becomes imaginary, $x = iy$, and the equation then reads

$$\frac{2y}{a} = \tanh y \tag{16}$$

which, incidentally, is the mean-field equation for a spin one-half ferromagnet. Substituting the solutions into

$$\gamma(q)=\zeta\sqrt{q^2+\left(\frac{a}{2L}\right)^2} \tag{17}$$

then gives a result which is again proportional to $1/L$. The local magnetic surface exponent $\beta_{l,1}$ then follows from $L/\xi=\zeta\pi\beta_{l,1}$. In general, it has to be calculated numerically, but limiting forms are

$$\beta_{l,1}(a)=\begin{cases}\frac{1}{2}-(a/\pi^2)+\mathrm{O}(a^2) & |a|\ll 1\\ (1/\pi)a\,\mathrm{e}^{-a/2}+\mathrm{O}(\mathrm{e}^{-a}) & a\gg 1\\ (|a|/2\pi)+(\pi/|a|)+\mathrm{O}(a^{-2}) & a\ll -1.\end{cases} \tag{18}$$

The complete function $\beta_{l,1}(a)$ is also shown in figure 1. The phenomenological theory has not yet been worked out for the present case, so the result for $\beta_{l,1}$ is completely new.

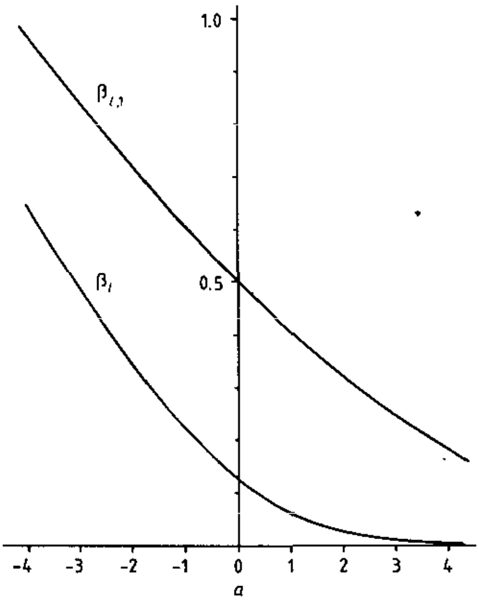

Figure 1. Local critical exponent of the magnetization near a defect situated in the interior (β_l) and on the boundary ($\beta_{l,1}$) as a function of the microscopic parameter a, equation (10).

In conclusion we have shown that, for a defect with radial symmetry, conformal invariance holds and the local magnetic exponent is a function of the microscopic parameters and thus non-universal. Its qualitative behaviour is the same for a defect in the interior or on the boundary. It becomes smaller if the defect enhances the couplings over their bulk value. This reflects the tendency towards local order and is similar to the case of a system with an inhomogeneous straight boundary [8-11]. However, in contrast to this latter case, the exponent does not vanish here for any finite value of A. Therefore one does not find a local magnetization which stays finite right up to the critical temperature and then drops to zero discontinuously. On the other hand, if the defect diminishes the couplings, the same linear dependence of the exponents on A is found here for large negative values of A.

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